Integer right triangles, the square root of two, adjacent numbers and relationships.

by Mel Cox

What follows is the result of some recreational math. It started simply from wondering how to find sets of three integers (whole numbers) that satisfied the Pythagorean theorem: $a^2+b^2=c^2$. It wound up in an incredible (to me) gird lock of very simple equations, some cute algorithms, all the integer triangles, even some easily found approximations of square roots of integers. Nothing profound, but fun! It might inspire readers to attack some recreational math for the fun and surprises. (Warning: there is no demanding algebra in this, no computer is required; a calculator might be convenient in some places, and you don't even have to extract any square roots.)

So how can you generate *integer* solutions to $a^2+b^2=c^2$? (I have since discovered that there must be dozens of ways, but this was *my* challenge.)

Here's how:

$a^2+b^2=c^2$ (a, b, and c all integers)

$a^2=c^2-b^2 = (c+b)(c-b)$ and here is a key: $(c+b)$ and $(c-b)$ are complimentary factors of a^2 and recognize that if you know $(c+b)$ and $(c-b)$ then you can find c and b because

$(c+b) + (c-b) = 2c$ and $(c+b) - (c-b) = 2b$.

let $c+b = s$ (for sum)

$c-b = d$ (for difference)

thus:

$$\left\{\begin{array}{c} a^2=sd \\ b=\frac{1}{2}(s-d) \\ c=\frac{1}{2}(s+d) \end{array}\right\} !$$

(Equation Set 1)

Here is a recipe for proceeding: select an integer a (to be one side of your right triangle) and write down all its factors twice, then allocate all of these to s and d.

for example:

$$a = 15$$

$$a^2 = 1\cdot1\cdot3\cdot3\cdot5\cdot5$$

let $$s = 1\cdot5\cdot5 = 25$$

$$d = 1\cdot3\cdot3 = 9$$

$$b = \frac{1}{2}(25-9) = 8$$

$$c = \frac{1}{2}(25+9) = 17$$

so the triangle a, b, c becomes (abbreviated) $\Delta\ a,b,c = 15, 8, 17$

The reader is encouraged to distribute these factors in other ways (even exhaust all the ways) and find other triangles with 15 as one side. Don't ignore the case $s = 1\cdot5\cdot5\cdot3\cdot3$, $d = 1$.

If you do that, you will better understand why some constraints are desirable. if $s = d$, b=0 and $c = a$, a perfectly legitimate solution to $a^2+b^2=c^2$, but trivial.

If $s < d$ (s smaller than d), then b comes out negative, also perfectly legitimate in a solution to $a^2+b^2=c^2$, but it's trivial since if you have a solution consisting of positive integers, you can obtain all the solutions containing negative integers by just changing the signs in any combination you wish.

If s and d have a common factor (of necessity a factor of a) then it propagates through the arithmetic to become a factor of b and c. (In the example above, let $s = 1\cdot5\cdot5\cdot3$, $d = 1\cdot3$. You obtain $\Delta\ a,b,c = 15, 36, 39$). Divide out the common factor to obtain a **Primitive Integer Right Triangle (PIRT).** The non primitives (containing the common factor) are also trivial, really poor second-class cousins, since they can be generated in profusion from any PIRT. Let's stick to PIRTs.

If a is selected an odd number (as 15 in the example) distribute the factors of a^2 such that:

For a an odd integer

1) $s > d$

2) s and d have no common factors

You are encouraged to get your scratch pad and pencil again and make triangles from side a an even integer and the we will look at some other constraints. If you do this, do observe the constraint $s > d$, for the same reason as before.

Assuming you scratched some, you will appreciate the constraint:

For a an *even* integer:
$s > d$
s and d must be even, but one or the other must contain only a single factor of 2, and s and d to have no other common factors.

and: a must be a multiple of 4

You are now equipped to make the following observations about Primitive Integer Right Triangles:

- one side is odd, the other even, the hypotenuse odd
- the even side is a multiple of 4
- the *sum* of the hypotenuse and the even side is a perfect square of an odd integer.
- the *difference* between the hypotenuse and the odd side is twice a perfect square.
- all odd integers can participate as one side of at least one **PIRT**.

You may also note that if you use the technique defined above to find all the PIRTs starting with odd numbers, no new PIRTs are to be found by starting with even numbers.

Fun for a moment: Algorithms are strange, mysterious and appealing because they disguise the underlying math. Here are two beauties that you can use to show

your mastery of integer right triangles (even to some accomplished mathematicians).

"Give me a number and I will give you an integer right triangle with it as one side".

If odd: square it; divide by two; pick the adjacent integers a s the other side and the hypotenuse.
(example: given 9: take $81 \div 2 = 40\ 1/2$; use 40 and 41)

If even; divide by 2; square this; pick the adjacent integers as the other side and the hypotenuse.
(example: given 8: take $8 \div 2 = 4$; $4^2=16$; use 15 and 17)

Long before all of the above fell into place, a second challenge came up. If the two sides of an integer triangle were nearly equal, then the hypotenuse would be nearly $\sqrt{2}$ times one of the sides as is the precise case in an equilateral right triangle. The 3, 4, 5 right triangle is a case in point. 3 and 4 are as close to being equal as two different integers can be. 5/3 is larger than $\sqrt{2}$ and 5/4 is smaller — Aha, what about the average of 3 and 4? Could it be that

$$\frac{3+4}{2}\sqrt{2} \approx 5 \text{ or } \frac{3+4}{\sqrt{2}} \approx 5$$

or $\sqrt{2} \approx \frac{7}{5} = 1.4$ Pretty good! ($\sqrt{2} = 1.414$---) If there were a larger triangle with sides differing by only 1, the 1 would be less significant and the approximation better.

A little poking around amongst the numbers and the triangle 20, 21, 29 fell out. As before:

$$\sqrt{2} \cong \frac{41}{29} = 1.4138 \quad (\sqrt{2}=1.4142)$$

Yep! Better.
Notice that the fractions

$\left(\frac{7}{5},\frac{41}{29}\right)$ when squared must approximate 2.

$$\left(\frac{7}{5}\right)^2=\left(\frac{49}{25}\right)\approx 2 \text{ and } \left(\frac{41}{29}\right)=\frac{1681}{841}\approx 2.$$

And now observe that

$$\left(\frac{49}{25}\right)\approx 2\;;\;\; 2(25)=50=49+1$$

and $$\left(\frac{1681}{841}\right)\approx 2\;;\;\; 2(841)=1682=1681+1$$

49 and 50 , as well as 1681 and 1682 are *adjacent* numbers where one is a *perfect square* and the other is *twice a perfect square*. You recognize that these numbers cannot be equal; if they were you would have a fraction that was exactly $\sqrt{2}$ and we know that $\sqrt{2}$ is irrational. But two different integers can't be more equal than to differ by only one. So for size of the integers in the approximations they can't be any better. If for example $\left(\frac{41}{29}\right)\approx\sqrt{2}$ then $\left(\frac{40}{28}\right)$ is also $\approx\sqrt{2}$ but

$\left(\frac{40}{28}\right)^2=\frac{1600}{784}$ and $2(784)=1568=1600-32$ and so $\frac{41}{29}$ is the much better approximation of $\sqrt{2}$.

A little more hunting and pecking (trying to find adjacent numbers, one a square, the other twice a square) and the following sequence of fractions which approximate the $\sqrt{2}$ was at hand:

$$\frac{3}{2},\frac{7}{5},\frac{17}{12},\frac{41}{29},\frac{99}{70}$$

At least one pattern of this sequence is almost obvious: the next numerator is twice the present numerator plus the previous—same for denominators (e.g. $17+17+7=41$ and $12+12+5=29$).

You might also notice that the ratio of a numerator (or denominator) to its predecessor (eg 41 : 17) changes from fraction but seems to be converting on a number near 2.4 — and you might speculate that this is $\sqrt{2}+1$, when you

look at is reciprocal (the ratio of the lower number to the higher, (e.g. 17 : 14) and see that it is $\sqrt{2}-1$; and $x+1 = 1/(x-1)$ and see the $\sqrt{2}$ unique; you might speculate that this *must* be true because of the way all these other numbers are locked together.

This series can be continued to as large a pair of numbers as one likes. Every other pair of numbers identifies an integer right triangle where $b = a + 1$:

$\frac{7}{5}$: 3, 4, 5

$\frac{41}{29}$: 20, 21, 29

$\frac{239}{169}$: 119, 120, 169

Each triplet is a PIRT, *a,b,c.*

The pairs in between these identify other number patterns:

$\frac{3}{2}$: 1, 2, 2

$\frac{17}{12}$: 8, 9, 12

$\frac{99}{70}$: 49, 50, 70

These triplets are not right triangles but curiously, $\hat{a}^2+\hat{b}^2=\hat{c}^2+1$

$$1^2+2^2=2^2+1$$
$$8^2+9^2=12^2+1$$
$$49^2+50^2=70^2+1$$

We have an algorithm for generating the desired triangles and the approximations for $\sqrt{2}$ but we don't know the underlying math. And as we've said before algorithms are strange, mysterious and appealing—still a challenge.

I didn't want to "hunt and peck"; I wanted to be analytical. Let's restate some earlier equations

$$a^2=sd$$
$$b=\frac{1}{2}(s-d)$$
$$c=\frac{1}{2}(s+d) \quad \textit{(Equation Set 1)}$$

Setting $b = a+1$, and substituting $\frac{a^2}{d}$ for s, we obtain

$$a+1=\frac{1}{2}\left(\frac{a^2}{d}-d\right)$$

This "simplifies" to $a^2-2ad-2d-d^2=0$

For this equation every value of a generates two values of d, and every value of d generates two values of a. For example we know that for the 3,4,5 triangle, a=3, d=1. Now put 3 into the equation and eventually:

$$d^2 + 8d - 9 = 0$$
$$(d+9)(d-1) = 0$$
$$d = 1, -9$$

Now substitute the -9 as d into the set (1) and you obtain the PIRT: 3, 4, -5. Indeed a different PIRT (c = -5, not +5) and $b = a + 1$.

Now substitute d= -9 into the quadratic (watch those minus signs!), simplify and obtain

$$a^2 + 18a - 63 = 0$$
$$(a+21)(a-3) = 0$$
$$a = -21, 3.$$

The three in the a from which we obtained d = -9, gives us a = -21. Again substitute -21 into set (1) and obtain a new PIRT: -21, -20, -29. You can drop the minus signs *and*

switch a & b; and you have 20, 21, 29 and still $b = a + 1$. Now you can use the 20 to get a new d, and the new d to get a new a.

Now you don't have to hunt and peck for the larger PIRTs (b = a + 1) and you thus have alternate values of approximate $\sqrt{2}$ based on PIRTs. But those intervening numbers where $\hat{a}^2+\hat{b}^2=\hat{c}^2+1$— still a mystery.

It isn't clear how the next two equations came about. It is almost easy to stumble onto them—perhaps as follows: In a given PIRT there are two Ss: one *a perfect square* the other *twice a perfect square*. Take the case of the PIRT: 3, 4, 5.

$$s_a = c + a = 5 + 3 = 8 = 2 \cdot 2^2 \text{ and}$$

$$s_b = c + b = 5 + 4 = 9 = 3^2$$

Just plain out of curiosity note their product $s_a s_b = 2 \cdot 2^2 \cdot 3^2$ whose square root you can write down by inspection: $2 \cdot 2 \cdot 3 = 12$, "coincidentally" $(a + b + c)$. You try this on another triangle and it fits there as well, so here it is as a theorem:

Let $(a + b + c) = P$ (for Perimeter), then $P^2 = 2s_a s_b$ (Let's call this the "perimeter equation"). Substitute in the a b and c:

$$(a + b + c)^2 = 2(c + a)(c + b)$$

do the squaring and multiplying and lo, $a^2 + b^2 = c^2$, so the perimeter equation is merely the Pythagorean equation in different terms. Now recall that in a given PIRT there are 2 ds; one *a perfect square*, the other *twice a perfect square*. For the 3, 4, 5:

$$d_a = 5 - 3 = 2 = 2 \cdot 1^2$$
$$d_b = 5 - 4 = 1 = 1^2$$

as before $d_a d_b = (2 \cdot 1^2)(1^2)$. Multiply by 2: $2 \cdot 2 \cdot 1^2 \cdot 1^2$ whose square root is 2

"coincidentally" $(a + b - c)$. So,

Theorem: let $(a + b - c) = D$ (for big difference).

$D^2 = 2d_ad_b$ (the "difference equation")

Again substitute and simplify and this to is a restatement of $^2 + b^2 = c^2$ in other terms. Write the two equations:

$$P^2 = 2s_as_b$$

$$D^2 = 2d_ad_b$$

and obviously the set of integers that fit one equation also fit the other.

Notion: Take the numbers (P, s_a, s_b) from one triangle and call them the numbers (D, d_a, d_b) of a new triangle and solve for the new triangle a', b', c'. (Also notice that the difference between a & b is preserved in s_a & s_b and d_a & d_b and a' & b'). If you do this a couple of times you might want to do some substituting and simplifying to obtain the

<u>Progression Algorithm</u>	<u>Ex: $a, b, c = 3, 4, 5$ ($P = 12$)</u>	
$a' = 2P - b$	$a' = 2 \cdot 12 - 4$	$= 20$
$b' = 2P - a$	$b' = 24 - 3$	$= 21$
$c' = 2P + c$	$c' = 24 + 5$	$= 29$

<u>Regression Algorithm</u>	<u>Ex: $a, b, c = 8, 15, 17$ ($D = 6$)</u>
$a^* = 2D - b$	$a^* = 12 - 15 = -3$
$b^* = 2D - a$	$b^* = 12 - 8 = 4$
$c^* = c - 2D$	$c^* = 17 - 12 = 5$

Take another look at $P^2 = 2s_as_b$ for the cases $b = a + 1$ and that of s_b and s_a one is a *perfect square* and the other *twice a perfect square* and they differ by one.

$$P = \sqrt{2s_as_b}$$

$$P = \sqrt{2}\sqrt{s_a s_b}$$

Since s_a & s_b differ by only 1, their average is a good approximation of $\sqrt{s_a s_b}$, i.e.

$$\frac{s_a+s_b}{2} \approx \sqrt{s_a s_b}$$

$$P \approx \sqrt{2}\frac{s_a+s_b}{2} \approx \frac{s_a+s_b}{2}$$

$$\sqrt{2} \approx \frac{s_a+s_b}{P}$$

From the Δ $a,b,c = 3,4,5$; $P = 12$, $s_a = 8$, $s_b = 9$;

$$\sqrt{2} \approx \frac{17}{12}$$

from Δ $a,b,c = 20, 21, 29$; $P = 70$, $s_a = 49$, $s_b = 50$;

$$\sqrt{2} \approx \frac{99}{70}$$

and these are the mystery ratios from the approximate $\sqrt{2}$ series, now also triangle based. If you "plug" the a,b,c integers into the Perimeter equation you will also note $2 \cdot 3 \cdot 4 = 5^2 - 1$ deja vu!

We might still be a little dissatisfied that we are iterating to obtain the larger numbers in the approximate $\sqrt{2}$ series. Might we not take that expression ($\sqrt{2}$ +1) which we suspected was the multiplier of one number to the next, multiply it by itself several times and let it help us take a giant step instead of iterating?

$$\begin{array}{ll} & \sqrt{2} + 1 \\ M(\sqrt{2} + 1) & \sqrt{2} + 1 \\ \hline & 2 + 2\sqrt{2} + 1 = 2\sqrt{2} + 3 \end{array}$$

$$\begin{array}{ll} & 2\sqrt{2} + 3 \\ M(\sqrt{2} + 1) & \sqrt{2} + 1 \\ \hline & 4 + 5\sqrt{2} + 3 = 5\sqrt{2} + 7 \end{array}$$

$$\begin{array}{ll} & 5\sqrt{2} + 7 \\ M(\sqrt{2} + 1) & \sqrt{2} + 1 \\ \hline & 10 + 12\sqrt{2} + 7 = 12\sqrt{2} + 17 \end{array}$$

$$\begin{array}{ll} & 12\sqrt{2} + 17 \\ M(\sqrt{2} + 1) & \sqrt{2} + 1 \\ \hline & 24 + 29\sqrt{2} + 17 = 29\sqrt{2} + 41 \end{array}$$

Surely you've noticed that the integers form the fractions that are the series approximating $\sqrt{2}$:

$$\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$$

Not just a "help" in generating the, *THE SERIES ITSELF! Mathematical Serendipity!*

Why? Remember the reciprocal of $\sqrt{2}$ + 1 being $\sqrt{2}$ - 1? $\sqrt{2}$ -1 is less that 1 and when multiplied by itself becomes smaller and smaller (approaches zero). The plus sign

changes to a minus sign so that one of these becomes e.g. $12\sqrt{2} - 17$ and of which we can say

$$12\sqrt{2} - 17 \approx 0$$

and $\sqrt{2} \approx \frac{17}{12}$ etc.

So we can take the expression $(\sqrt{2} - 1)^n$, break out Pascals triangle to get the coefficients of the binomial expansion and obtain very large integers and very accurate approximations of the $\sqrt{2}$. Similarly $(2-\sqrt{3})$, $(\sqrt{5} - 2)$, $(3 - \sqrt{7})$, etc.!

The book now closes on this episode of mathematical recreation, there have been immense amounts of pleasure and self satisfaction along the way.